QM Homework 1: Solutions

1. (a) $[A,B] \equiv AB - BA = -(BA - AB) \equiv -[B,A]$

   (b) $$[A+B,C] \equiv (A+B)C - C(A+B) = AC + BC - CA - CB$$
   $$= (AC - CA) + (BC - CB) \equiv [A,C] + [B,C]$$

   (c) $$[AB,C] \equiv ABC - CAB = ABC - ACB + ACB - CAB$$
   $$= A(BC - CB) + (AC - CA)B \equiv A[B,C] + [A,C]B$$

   (d) $$[A^{-1},B] \equiv A^{-1}B - BA^{-1} = A^{-1}BAA^{-1} - A^{-1}ABA^{-1}$$
   $$= A^{-1}(BA - AB)A^{-1} \equiv -A^{-1}[A,B]A^{-1}$$

   (e) $$[[A,B],C] + [[B,C],A] + [[C,A],B]$$
   $$\equiv [AB - BA, C] + [BC - CB, A] + [CA - AC, B]$$
   $$\begin{aligned} &\equiv (AB - BA)C - C(AB - BA) \\ &+(BC - CB)A - A(BC - CB) \\ &+(CA - AC)B - B(CA - AC) \end{aligned}$$
   $$\begin{aligned} &= ABC - BAC - CAB + CBA \\ &+BCA - CBA - ABC + ACB \\ &+CAB - ACB - BCA + BAC \end{aligned}$$
   $$\equiv 0$$

2. First note that for any function $f(x)$ that has a Taylor expansion about $x = 0$ we can define a corresponding function $f(A)$ of an operator A:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{dx^n} \right) x^n \qquad \Rightarrow \qquad f(A) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{dx^n} \right) A^n$$

Then:

$$e^{\lambda A} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A^n$$

$$\frac{d}{d\lambda} e^{\lambda A} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} A^n = A \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A^{n-1} = Ae^{\lambda A}$$

However, we may use $A^n = A^{n-1}A$ instead of $A^n = AA^{n-1}$ in each term of the last sum and equivalently conclude:

$$\frac{d}{d\lambda} e^{\lambda A} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} A^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} A^{n-1}A = e^{\lambda A} A$$

Clearly, A commutes with $e^{\lambda A}$, and likewise with any other expandable function $f(A)$. Now define a function $F(\lambda) = e^{\lambda A} B e^{-\lambda A}$ of a scalar λ and take its derivative with respect to λ:

$$\frac{d}{d\lambda} e^{\lambda A} B e^{-\lambda A} = \left(e^{\lambda A} A \right) B a^{-\lambda A} + e^{\lambda A} B \left(-A a^{-\lambda A} \right) = e^{\lambda A} [A,B] e^{-\lambda A}$$

This is a formal rule that we can apply to any operator, not just B. So, we apply it to B, then $[A, B]$, etc. to take various derivatives of $F(\lambda)$:

$$\begin{aligned}
\frac{dF}{d\lambda} &= \frac{d}{d\lambda} e^{\lambda A} B e^{-\lambda A} = e^{\lambda A}[A, B]e^{-\lambda A} \\
\frac{d^2F}{d\lambda^2} &= \frac{d}{d\lambda} e^{\lambda A}[A, B]e^{-\lambda A} = e^{\lambda A}[A, [A, B]]e^{-\lambda A} \\
\frac{d^3F}{d\lambda^3} &= \frac{d}{d\lambda} e^{\lambda A}[A, [A, B]]e^{-\lambda A} = e^{\lambda A}[A, [A, [A, B]]]e^{-\lambda A} \\
&\cdots
\end{aligned}$$

By (mathematical) induction we have:

$$\frac{d^nF}{d\lambda^n} = e^{\lambda A}[A, [A, \ldots [A, B] \cdots]]e^{-\lambda A}$$

with n appearences of commutators and the operator A on the right-hand side. The Taylor expansion of $F(\lambda)$ is therefore:

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^nF(0)}{d\lambda^n} \right) \lambda^n = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, \ldots [A, B] \cdots]] \lambda^n$$

Substituting $\lambda = 1$ here proves the sought formula.

3. (a) Use the eigenstates (labeled $|a\rangle$ or $|b\rangle$) of any Hermitian operator to express the trace, and then insert the identity operator constructed from the same eigenstates:

$$\begin{aligned}
\mathrm{tr}(XY) &\equiv \sum_a \langle a|XY|a\rangle = \sum_a \sum_b \langle a|X|b\rangle\langle b|Y|a\rangle = \sum_a \sum_b \langle b|Y|a\rangle\langle a|X|b\rangle \\
&= \sum_b \langle b|YX|b\rangle \equiv \mathrm{tr}(YX)
\end{aligned}$$

(b) Similarly:

$$\mathrm{tr}(|\psi\rangle\langle\phi|) \equiv \sum_a \langle a|\psi\rangle\langle\phi|a\rangle = \sum_a \langle\phi|a\rangle\langle a|\psi\rangle \equiv \langle\phi|\psi\rangle$$

(c) Using:

$$A \equiv \sum_a |a\rangle a \langle a|$$

and the Taylor expansions:

$$\begin{aligned}
f(a) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) a^n \\
f(A) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) A^n
\end{aligned}$$

we find:

$$\begin{aligned}
f(A) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) \left[\sum_a |a\rangle a \langle a| \right]^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) \left[\sum_{a_1} |a_1\rangle a_1 \langle a_1| \right] \left[\sum_{a_2} |a_2\rangle a_2 \langle a_2| \right] \cdots \left[\sum_{a_n} |a_n\rangle a_n \langle a_n| \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) \sum_{a_1} \cdots \sum_{a_n} |a_1\rangle a_1 \langle a_1 | a_2 \rangle a_2 \langle a_2 | \cdots |a_{n-1}\rangle a_{n-1} \langle a_{n-1} | a_n \rangle a_n \langle a_n | \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) \sum_{a} |a\rangle a^n \langle a| \\
&= \sum_{a} |a\rangle \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(0)}{da^n} \right) a^n \right] \langle a| \\
&= \sum_{a} |a\rangle f(a) \langle a|
\end{aligned}
$$

Note that taking a sum (over a) to n^{th} power requires introducing n independent summation labels $a_1, \dots, a_n$. However, all of them end up being equal in the end by orthogonality of the operator A's eigenstates. The remaining sum over $a \equiv a_1 = \cdots = a_n$ commutes with the sum over n.

4. (a) This is slightly tedious but straight-forward: requires computing nine products of 2×2 matrices.

(b) First note that:

$$
\begin{aligned}
(\boldsymbol{\sigma}\hat{\mathbf{n}})^0 &\equiv 1 \\
(\boldsymbol{\sigma}\hat{\mathbf{n}})^1 &\equiv \boldsymbol{\sigma}\hat{\mathbf{n}} = \sigma_i \hat{n}_i \\
(\boldsymbol{\sigma}\hat{\mathbf{n}})^2 &\equiv (\boldsymbol{\sigma}\hat{\mathbf{n}})(\boldsymbol{\sigma}\hat{\mathbf{n}}) = (\sigma_i \hat{n}_i)(\sigma_j \hat{n}_j) = \hat{n}_i \hat{n}_j (\sigma_i \sigma_j) = \hat{n}_i \hat{n}_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k) \\
&= \hat{n}_i \hat{n}_i + i\epsilon_{ijk} \hat{n}_i \hat{n}_j \sigma_k = 1
\end{aligned}
$$

We used Einsten's notation and the result of part (a) in the last line (don't confuse the imaginary unit i with the summation index i). The Kronecker symbol δ_{ij} produces the sum $\hat{n}_i \hat{n}_i$ which is nothing but the norm of the unit-vector $|\hat{\mathbf{n}}| = 1$. The second term involving the Levi-Civita tensor is zero because the unit-vector components $\hat{n}_i$ and $\hat{n}_j$ are ordinary commuting numbers while $\epsilon_{ijk} = -\epsilon_{jik}$ (and i, j are summed over). Since $(\boldsymbol{\sigma}\hat{\mathbf{n}})^2 = 1$, we inductively find:

$$
(\boldsymbol{\sigma}\hat{\mathbf{n}})^k = \left\{ \begin{array}{ccc} 1 & , & \text{even } k \\ (\boldsymbol{\sigma}\hat{\mathbf{n}})^2 & , & \text{odd } k \end{array} \right\}
$$

Now Taylor-expand the left-hand side of the formula we want to prove:

$$
\begin{aligned}
e^{i\boldsymbol{\sigma}\hat{\mathbf{n}}\theta} &= \sum_{n=0}^{\infty} \frac{1}{n!} (i\boldsymbol{\sigma}\hat{\mathbf{n}}\theta)^n = \sum_{\text{even } n} \frac{(i\theta)^n}{n!} (\boldsymbol{\sigma}\hat{\mathbf{n}})^n + \sum_{\text{odd } n} \frac{(i\theta)^n}{n!} (\boldsymbol{\sigma}\hat{\mathbf{n}})^n \\
&= \sum_{\text{even } n} \frac{(i\theta)^n}{n!} + \sum_{\text{odd } n} \frac{(i\theta)^n}{n!} (\boldsymbol{\sigma}\hat{\mathbf{n}}) \\
&= \cosh(i\theta) + (\boldsymbol{\sigma}\hat{\mathbf{n}}) \sinh(i\theta) \\
&= \cos(\theta) + i(\boldsymbol{\sigma}\hat{\mathbf{n}}) \sin(\theta)
\end{aligned}
$$

In the last step we recognized the Taylor expansions of the hyperbolic cosine and sine, and related them to the ordinary cosine and sine functions.