QM Homework 1: Fundamental Concepts

1. Prove from the definition of commutators that the following expressions hold for any operators A, B, C:
   (a) $[A, B] = -[B, A]$
   (b) $[A + B, C] = [A, C] + [B, C]$
   (c) $[AB, C] = A[B, C] + [A, C]B$
   (d) $[A^{-1}, B] = -A^{-1}[A, B]A^{-1}$
   (e) $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

2. Derive the formula:
$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!}[A, [A, \ldots [A, B] \cdots]]$$
where the n^{th} term on the right-hand side has n commutators (and n appearences of the operator A.

   Use the following approach: Consider the operator $F(\lambda) = e^{\lambda A} B e^{-\lambda A}$ that depends on a number λ. Expand $F(\lambda)$ in a Taylor series by differentiating it repeatedly with respect to λ. Pay attention to the fact that A and B generally do not commute. Eventually set $\lambda = 1$.

3. Using the rules of bra-ket algebra, prove the following for arbitrary operators X, Y, a Hermitian operator A whose eigenstates and corresponding eigenvalues are $|a\rangle$ and a respectively, and arbitrary states $|\psi\rangle, |\phi\rangle$:
   (a) $\text{tr}(XY) = \text{tr}(YX)$
   (b) $\text{tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$
   (c) $f(A) = \sum_a |a\rangle f(a) \langle a|$ (hint: use Taylor expansion to define a function of an operator)

4. (a) Prove that:
$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$
where $i, j, k \in \{x, y, z\}$, δ_{ij} is the Kronecker symbol (1 if its indices are equal, otherwise 0), ϵ_{ijk} is the Levi-Civita symbol (0 if any two indices are equal, changes sign if any two indices exchange place, $\epsilon_{xyz} \equiv 1$), and
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
are Pauli matrices. We are using Einstein's notation: the repeated index k on the right-hand side is summed over all of its possible values.

   (b) Then show that:
$$\exp(i\boldsymbol{\sigma}\hat{\mathbf{n}}\theta) = \cos\theta + i\boldsymbol{\sigma}\hat{\mathbf{n}}\sin(\theta)$$

for any angle θ and unit-vector $\hat{\mathbf{n}}$ in real space. $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices.

5. A spin $S = \frac{1}{2}$ system is known to be in an eigenstate of $\mathbf{S}\hat{\mathbf{n}}$ with eigenvalue $+\frac{\hbar}{2}$, where $\hat{\mathbf{n}}$ is a unit-vector lying in the xz-plane that makes an angle γ with the positive z-axis.

   (a) Suppose S_x is measured. What is the probability of getting $+\frac{\hbar}{2}$?

   (b) Evaluate the dispersion of S_x, i.e.:
   $$\langle (S_x - \langle S_x \rangle)^2 \rangle$$
   Check the special cases for $\gamma = 0$, $\gamma = \frac{\pi}{2}$ and $\gamma = \pi$.

Other suggested problems: Sakurai (2nd edition) 1.6, 1.9, 1.13, 1.15, 1.16, 1.21, 1.23, 1.26, 1.33